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Axiomatizing Schemes and Their Behaviors

STEPHEN L. BLOOM AND ZOLTÁN ESIK*

Department of Pure and Applied Mathematics, Stevens Institute of Technology, Hoboken, New Jersey 07030

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INTRODUCTION

The syntax and the semantics of flowchart algorithms may be conveniently separated. The syntax of a flowchart algorithm is the underlying flowchart scheme. The semantics is given by an interpretation of each symbol appearing on the nodes of the scheme. One aspect of this theory is the determination of when two schemes have the same "behavior" under all interpretations. When "behavior" is interpreted in the strongest sense as "performing the same stepwise computations," the collection of equivalence classes of schemes was shown by Elgot [E-MC] to have the structure of an "iterative theory." The strong behavior of a scheme is precisely captured by the (labeled, locally ordered, locally finite) tree obtained by "unfolding" the scheme in the familiar way. (When the "behavior" of a scheme F is considered to be the partial input–output function computed by F in each interpretation, certain quotients of the trees are the appropriate interpretations.)

The collection of labeled schemes and trees is equipped with the same structure, which is preserved by the map which takes a scheme F to its strong behavior $[F]$. This structure is simple to describe: starting from the atomic and "base" schemes, using just three operations (composition, tupling, and iteration), any scheme may be constructed. The operation of iteration (or "looping") on schemes and trees was not defined in all cases in the setup of [E-MC]; for example, consider the scheme $1 \to 1$ consisting of only an exit (and the tree which forms its strong behavior). Thus the schemes and trees form a partial algebra. It was to remedy this situation on the semantic algebras of trees and other iterative theories that the notion of an "iteration" theory was introduced (in [BEW]). Iteration theories are (total) algebras with the same operations as iterative theories.

The collection of trees which are strong behaviors of schemes may be given the structure of an iteration theory by completing the iteration operation in an almost arbitrary way. The collection of iteration theories is the class of all models of the set of equations valid in all iteration theories of trees. (See [BEW] for details.)

* Permanent address: A. Joseph University, Bolyai Institute, Szeged, Hungary.

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However, the schemes themselves do not have the structure of an iteration theory: the looping operation is not total.

In the present paper, we "complete" the story of iteration theories in the following sense. We define, in an intuitively natural way, a total iteration operation on the class of schemes such that the unfolding map $F \to [F]$ preserves iteration (as well as the other operations). We then classify the resulting (many sorted) algebra of schemes as a free algebra in a certain equational class K_0. In fact, more constructively, we find a finite set of equational axiom schemes A_0 for the class K_0. Secondly, we show that the class K_0 is the same whether we take the class of schemes to be generated by "scalar atomic schemes" (i.e., schemes with one input edge and perhaps several output edges) or whether we take the class of schemes to be generated by vector atomic schemes (which may have several input, as well as output edges). Third, we show that by adding two additional axioms to A_0, we obtain an axiomatization of the class K of iteration theories themselves. The main point here is not the axiomatization of iteration theories (which was found earlier by Esik in [Es]). Instead, the merit of the present argument is that it yields an effective method of generating a proof in equational logic of any valid equation

$$\text{“}t = t'\text{”}$$

between "iteration terms." The method also yields a new algorithm to decide whether a given equation is valid.

There has been some previous work of a similar nature. An equational axiomatization for the class of reducible schemes was found by Elgot and Shepherdson in [ES]. An equational set of axioms for the "D-schemes" (or "while schemes") was given in [JY] and an equational axiomatization was found for their strong behaviors in [DT].

In order to keep the paper to an appropriate length, we will assume the reader has an acquaintance with the treatment of schemes given by Elgot in [E-SP], or by Elgot and Shepherdson [ES] and has some familiarity with at least one of the papers [E-MC], [BE], [EBT], [BEW], or [Es].

1. Algebras, Terms and Theories

All of the algebras we will be considering are $N \times N$-sorted (where N denotes the set of nonnegative integers); $A(n, p)$ (or $A_{n,p}$) denotes the underlying set of the algebra A of sort (n, p). We will write either $f \in A(n, p)$ or $f: n \to p$, whichever seems more convenient. Our algebras are equipped with the following operations and constants:

Composition. For each triple n, p, q of nonnegative integers, there is a binary operation of "composition"

$$\cdot : A(n, p) \times A(p, q) \to A(n, q)$$

$$f, g \mapsto f \cdot g.$$

Pairing. For each triple n, p, q of nonnegative integers, there is a binary operation "pairing"

$$\langle\ ,\ \rangle\colon A(n, q) \times A(p, q) \to A(n+p, q)$$
$$f, g \mapsto \langle f, g \rangle;$$

Constants. For each pair of nonnegative integers i, p with $i \in [p] = \{1, 2, ..., p\}$,

$$\pi_p^i \in A(1, p),$$

and for each nonnegative integer p, there is a constant

$$0_p \in A(0, p).$$

The class of all $N \times N$-sorted algebras with the above operations is denoted T_0. The class T_1 consists of all T_0 algebras enriched by the operation † of iteration, described as follows.

Iteration †: for each pair n, p of nonnegative integers

$$^\dagger\colon A(n, n+p) \to A(n, p)$$
$$f \mapsto f^\dagger.$$

In any T_1 algebra A, the element

$$(\pi_1^1)^\dagger\colon 1 \to 0$$

will be abbreviated $\perp$.

By a "doubly ranked set Σ" we understand a family of pairwise disjoint sets $\Sigma = (\Sigma(n, p)\colon (n, p) \in N \times N)$. (A "singly ranked set" Σ is a family Σ_n of pairwise disjoint sets indexed by N, and we will identify a singly ranked set Σ with a doubly ranked set Σ' having $\Sigma'(1, n) = \Sigma_n$ and $\Sigma'(p, q)$ empty, otherwise.) If A and B are doubly ranked sets, a rank-preserving $f\colon A \to B$ is a function which takes an element $a \in A(n, p)$ to $af \in B(n, p)$, for all pairs n, p. From now on, we will usually say only that f is a "function $A \to B$" instead of "rank-preserving function," when A and B are ranked sets.

If K is a class of T_0 or T_1 algebras and F is an algebra in K, we recall that "F is freely generated in K by Σ," more precisely, "freely generated in K by $\eta\colon \Sigma \to F$," where η is a (rank-preserving) function if for any (rank-preserving) function f from Σ to the underlying doubly ranked set of an algebra $A \in K$, then there is a unique homomorphism $f^\#\colon F \to A$ with $\eta \cdot f^\# = f$. In the case that K is the class of *all* algebras in the class T_i, $i = 0, 1$, the algebra freely generated by Σ in K is denoted "Σ-TM$_i$" (its elements will be called "terms") and the map η will be assumed to be an inclusion map.

If $f_i \in A(n_i, p)$, for $i = 1,..., k$, then we define their "source tupling"

$$\langle f_1,..., f_k \rangle : n_1 + \cdots + n_k \to p$$

by association to the left in case that $k > 1$; if $k = 1$, $\langle f \rangle = f$; if $k = 0$, $\langle \ \rangle = 0_p$.

Further, for each pair $n, p \in N \times N$ we define the constant $\perp_{n,p}$ by

$$\perp_{n,p} = \langle \perp \cdot 0_p,..., \perp \cdot 0_p \rangle,$$

where there are n elements in the source tupling.

Now for each partial function φ: $[n] \to [p]$, $(n, p) \in N \times N$, we associate the constant term

$$\varphi\hat{} = \langle f_1,..., f_n \rangle : n \to p,$$

where for each $i \in [n]$,

$$f_i = \pi_p^{i\varphi},$$

the distinguished constant term when $i\varphi$ is defined, and

$$f_i = \perp_{1,p}$$

otherwise. The term corresponding to the identity function $[n] \to [n]$ is denoted 1_n. For each n and p, the inclusion and translated inclusion functions κ and λ are defined by

$$\kappa: [n] \to [n+p]$$
$$i \mapsto i$$
$$\lambda: [p] \to [n+p]$$
$$i \mapsto n + i.$$

A term of the form $\varphi\hat{}$ is a "simple base term" (or a "simple total base term," if φ is a total function). The term $\varphi\hat{}$ corresponding to the function φ will sometimes be written without the superscript $\hat{}$.

If $f_i: n_i \to p_i$, $i = 1, 2$, are terms, then $f_1 + f_2 : n_1 + n_2 \to p_1 + p_2$ is an abbreviation for $\langle f_1 \cdot \kappa, f_2 \cdot \lambda \rangle$, where the target of both κ and λ is $p_1 + p_2$. For $n > 2$, the term $f_1 + \cdots + f_n$ is defined by association to the left.

The terms in Σ-TM_1 which are built from the constants π_p^i, 0_p, and $\perp$ using the operations of composition and pairing are called partial base terms. (Note that Σ plays no role in the definition of the partial base terms.) Those partial base terms which contain no occurrence of the symbol $\perp$ are called (total) base terms.

Since Σ-TM_i is freely generated by $\eta: \Sigma \to \Sigma$-TM_i, $i = 0, 1$, if $f: n \to p$ is any element of Σ-TM_i, f determines a "polynomial function" on each T_i algebra A,

$$f_A : (\Sigma \to A) \to A_{n,p},$$

in the usual way, where $(\Sigma \to A)$ denotes the set of rank-preserving maps from Σ to A. An equation

$$t = t'$$

between elements of $\Sigma\text{-TM}_i$ is well formed if, for some pair (n, p), both t and t' belong to $\Sigma\text{-TM}_i(n, p)$. Henceforth, all equations will be assumed well formed. An equation $t = t'$ is *valid* in an algebra A in T_i if $t_A = t'_A$. An equation is valid in a class K of algebras if it is valid in each algebra in the class.

We now fix a doubly ranked set Σ such that for each pair (n, p), the set $\Sigma(n, p)$ is infinite. The class of "algebraic theories" is defined to be the class of all T_0 algebras

$$A = (A_{n,p}, \cdot, \langle\ \ \rangle, \pi_p^i, 0_p)$$

in which the following equations between terms in $\Sigma\text{-TM}_0$ are valid:

$\text{TH}_1: f \cdot (g \cdot h) = (f \cdot g) \cdot h,$	all $f: n \to p,\ g: p \to q,\ h: q \to r.$
$\text{TH}_2:\ 1_n \cdot f = f = f \cdot 1_p,$	all $f: n \to p.$
$\text{TH}_3:\ \langle\langle f, g\rangle = \langle f, \langle g, h\rangle\rangle,$	all $f: m \to q,\ g: n \to q,\ h: p \to q.$
$\text{TH}_4:\ \langle f, 0_p\rangle = f = \langle 0_p, f\rangle,$	all $f: n \to p.$
$\text{TH}_5: \pi_n^i \cdot \langle f_1,\dots, f_n\rangle = f_i, f_1,\dots, f_n: 1 \to p,$	$i \in [n].$
$\text{TH}_6: \langle \pi_n^1 \cdot f,\dots, \pi_n^n \cdot f\rangle = f,$	all $f: n \to q.$

Let TH denote the class of all theories. Clearly, TH has all free algebras. Two theories play a special role: the theory F_0 which is the free algebra in TH freely generated by the empty set (i.e., the ranked set all of whose carriers are empty) and $F_\perp$, the theory freely generated by the ranked set Σ whose only nonempty carrier is $\Sigma_{1,0}$, which is the singleton set $\{\perp\}$. F_0 is isomorphic to the theory whose morphisms $n \to p$ are the total functions $[n] \to [p]$; composition in the theory is function composition, the constant π_p^i is the injection $[1] \to [p]$ with value i; $\langle f, g\rangle$ is just source tupling. Similarly, $F_\perp$ is (uniquely) isomorphic to the theory of all partial functions $[n] \to [p]$, for n, p in $N \times N$ where the constant $\perp$ corresponds to the empty function $[1] \to [0]$. For other examples of theories and indications of their use in the semantics of flowchart algorithms the reader should see [E-MC].

A "preiteration theory" is an algebra in T_1 which satisfies the identities TH_1–TH_6. Let PRE denote the class of all preiteration theories.

We now show that for preiteration theories, free algebras are generated by singly ranked sets.

Suppose that Σ is a doubly ranked set, F is a preiteration theory and that $\eta: \Sigma \to F$ is a (rank-preserving) map. Define the singly ranked set $\Sigma^\wedge$ by

$$\Sigma_p^\wedge = \bigcup\,(\Sigma_{n,p} \times [n]: n > 0),$$

and define the function $\eta\hat{}: \hat{\Sigma} \to F$ by

$$(\sigma, i)\, \eta\hat{} = \pi_n^i \cdot (\sigma\eta),$$

if $\sigma \in \Sigma_{n,p}$.

PROPOSITION 1.1. *For any class K of preiteration theories, F is freely generated by $\eta: \Sigma \to F$ iff F is freely generated by $\eta\hat{}: \hat{\Sigma} \to F$.*

Proof. Let $h: F \to A$ be any homomorphism. Then, for $\sigma \in \Sigma(n, p)$, $n \geq 0$,

$$\sigma\eta = \langle \pi_n^1 \cdot \sigma\eta, \ldots, \pi_n^n \cdot \sigma\eta \rangle,$$

and

$$\begin{aligned} \sigma\eta h &= \langle \pi_n^1 \cdot (\sigma\eta)\, h, \ldots, \pi_n^n \cdot (\sigma\eta)\, h \rangle, \\ &= \langle (\sigma, 1)\, \eta\hat{}h, \ldots, (\sigma, n)\, \eta\hat{}h \rangle. \end{aligned}$$

Thus h is determined by its values on the elements $\sigma\eta$ iff h is determined by its values on the elements $(\sigma, i)\, \eta\hat{}$.

Let K be an arbitrary class of preiteration theories.

COROLLARY 1.2. *If for each singly ranked set Σ, there is an algebra in K freely generated by Σ, then for every doubly ranked set Σ' there is an algebra in K freely generated by Σ'.*

The following corollary gives a sufficient condition for a set E of identities to be a set of axioms for the variety generated by K. (Below, a "scalar equation" is an equation between terms $1 \to p$, for some p.)

COROLLARY 1.3. *Let Σ be a doubly ranked set and suppose that E is a set of equations between elements of Σ-TM_1, containing the equations TH_1–TH_6, which are valid in K. Then, if any equation between scalar terms in a singly ranked set which is valid in K is derivable from E, then all equations valid in K will be derivable from E.*

Thus, to verify that a set E of identities (containing the theory axioms) is an axiomatization of the variety of preiteration theories generated by K, it is enough to verify that all scalar identities valid in K are derivable from E.

2. SCHEMES

In this section, we will use a slight modification of the algebra of schemes that was suggested in [ES, Sect. 6], and provide an equational axiomatization for this algebra. First, recall that in [E-SP] and elsewhere flowchart schemes were defined using "scalar atoms" in a given singly ranked set. We will define schemes with "vector atoms" in a given doubly ranked set Σ as follows (see [ES], where this possibility was suggested):

A Σ-flowchart scheme $F: n \to p$ consists of

(i) a finite set V of "vertices" and a finite set E of "edges"; a function from V to N, $v \mapsto v^{@}$, assigning to each vertex its "component count." Let $V^{@}$ denote the set of "signed vertices"

$$V^{@} = \{(v, i) | v \in V, i \in [v^{@}]\}.$$

(When $v^{@} = 1$, we identify v and the pair $(v, 1)$.) There is a "source" function $E \to V$ and a "target" function $E \to V^{@}$ (so a directed edge begins at a vertex and ends at some *component* of a vertex);

(ii) an injective function ex: $[p] = \{j \in N | 1 \leqslant j \leqslant p\}$ into the set of vertices of outdegree 0: the values of this injection are written $\mathrm{ex}_1, \ldots, \mathrm{ex}_p$; the vertex ex_i is the "ith exit vertex." There is a unique vertex marked $\perp$, the "loop vertex," distinct from any exit vertex; The component count of each exit and loop vertex is 1;

(iii) a function b from $[n]$ to the set $V^{@}$, the "begin function"; the value of b on $i \in [n]$ is written b_i or "begin i";

(iv) a labeling function which associates to each nonexit nonloop vertex v an element of $\Sigma(n, p)$, where $n = v^{@}$.

These data are subject to the following requirements: the labeling and source functions must be "compatible" in the sense that any nonexit vertex v is the source of exactly p edges if the label of v is in $\Sigma(n, p)$, for some n. These p edges are ordered so that one may speak of the ith edge with source v.

Associated with each element σ of $\Sigma(n, p)$ is an atomic scheme with $p + 2$ vertices. One may represent such an atomic scheme by the picture in Fig. 2.1.

We will define operations on the class of Σ-schemes making the schemes an algebra in T_1. The operations of composition and pairing are almost the same as those given in [E-SP]; our iteration operation is rather more complicated, due to the fact that it is total. (The reader familiar with [BGR] may wonder whether vec-

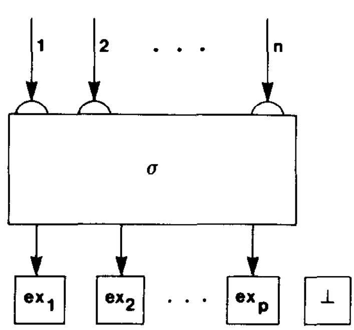

FIG. 2.1. $\sigma \in \Sigma(n, p)$ as an atomic scheme $n \to p$.

tor iteration may be replaced by scalar iteration. Unlike the case with iteration theories, vector iteration is necessary in the algebra of schemes. It may be shown that the class of schemes generated from the atomic and constant schemes using composition, pairing and scalar iteration is a proper subclass of the class of all schemes: indeed, each scheme in this class has the property that each cycle has a unique initial entry signed vertex. See [BT].)

Composition. The composite $F \cdot G$ of the schemes $F: n \to p$ and $G: p \to q$ is the scheme obtained by identifying the ith exit of F with the ith begin of G, and identifying the two loop vertices. The begins of $F \cdot G$ are those of F.

Pairing. The pairing $\langle F, G \rangle: n+m \to p$ of the schemes $F: n \to p$ and $G: m \to p$ is obtained by identifying the loop vertices and corresponding exits of F and G; no other vertices are identified; the first n begins are those of F; the last m begins are those of G.

Constants. If $i \in [p]$, then $\pi_p^i: 1 \to p$ is the scheme consisting only of p exit vertices and a loop vertex; the ith exit vertex is the begin. The scheme $0_p: 0 \to p$ is the same as π_p^i except for the fact that 0_p has no begin vertex.

Iteration. We might incorrectly define the iterate $F^\dagger: n \to p$ of the scheme $F: n \to n+p$ as follows: "identify each of the first n exits $\mathrm{ex}_1, \ldots, \mathrm{ex}_n$ of F with the corresponding begin; then relabel exit number $n+i$ as exit i, for $i \in [p]$." This definition does not make sense when some of the first n exits are themselves begins. A correct definition is more complicated. (We have chosen this definition of † so that the schemes with no vertex labeled by an element of Σ form an iteration theory isomorphic to the algebra of partial functions $[n] \to [p]$, $n, p \in N$.)

For any scheme $F: n \to n+p$, we define a "next vertex function,"

$$\nu: V^{(a)} \to V^{(a)}$$

as follows:

$$\text{if } (v, 1) = \mathrm{ex}_i,\ i \in [n], \text{ then } (v, 1)\,\nu = \mathrm{begin}_i \nu;$$

$$\text{otherwise,} \quad (v, i)\,\nu = (v, i).$$

This definition is recursive, and it may be explained as follows (recall that the component count of an exit is 1): for each exit ex_i, $i \in [n]$, form an "exit chain" $\mathrm{ex}_i = e_0, e_1, \ldots$; if $e_k = \mathrm{ex}_t$, and begin t is ex_j and $j \leqslant n$, then $e_{k+1} = \mathrm{ex}_j$; otherwise e_{k+1} is not defined, and the exit chain ends. For each $i \in [n]$, if it is possible to form an exit chain for ex_i of infinite length, ex_i is called "singular." (It is the singular exits for which the function ν is not defined.) If ex_i is not singular, the exit chain for ex_i ends, with ex_t say, because begin t is either a nonexit vertex or an exit with index at least $n+1$. In this case call begin $t (= \mathrm{ex}_i \nu)$ the "target" of the exit chain for ex_i.

We now define $F^\dagger$ as follows: for $i \in [n]$, attach the edges with target ex_i to the loop vertex if ex_i is singular, and otherwise attach the edges with target ex_i to $\mathrm{ex}_i v$; now delete the first n exits of F and relabel exit $n+j$ as exit j, for $j \in [p]$.

For each doubly ranked set Σ we have defined a T_1-algebra of schemes Σ-Sch, where $\Sigma\text{-Sch}(n, p)$ is the set of all Σ-schemes $n \to p$. We will identify isomorphic schemes, so that, for example, there is only one scheme $1 \to 0$ whose only vertex is the loop vertex labeled $\perp$.

DEFINITION 2.2. A term $t: n \to p$ in Σ-TM_1 is in "scheme normal form" (s.n.f.) if t is

$$\alpha \cdot \langle a^{\dagger}, 1_p, \perp_{1,p} \rangle,$$

where $\alpha: n \to m+p+1$ is a simple total base term, a is a source tupling $\langle a_1, ..., a_k \rangle$ and for each $i \in [k]$, a_i has the form

$$\sigma \cdot \varphi^{\wedge},$$

where $\sigma \in \Sigma(r, s)$, for some r, s and where $\varphi: s \to m+p$ is a partial function $[s] \to [m+p]$.

With Σ fixed, we denote by $|t|$ the value of a term t in Σ-TM_1 in the algebra Σ-Sch under the homomorphism determined by the function which takes $\sigma \in \Sigma(n, p)$ to the atomic scheme $\sigma: n \to p$.

The following theorem is due essentially to Elgot.

THEOREM 2.3 [E-SP, Theorem 3.1]. *For every Σ-scheme $F: n \to p$ there is a Σ-TM_1 term t in s.n.f. such that $|t| = F$.*

We want to be able to say when two s.n.f. terms denote the same scheme. In order to do so, we need some facts obtained in [ES].

Let s be the finite sequence $(n_1, ..., n_r)$ of nonnegative integers and suppose that $g: [r] \to [r]$ is a permutation. Let n be the sum of the numbers n_i.

DEFINITION 2.4. $g \# s: [n] \to [n]$ is the permutation which takes a number in $[n]$ of the form

$$n_1 + \cdots + n_k + j,$$

where $j \in [n_{k+1}]$, to the number $x + j$, where x is the sum of all numbers n_j such that $jg < (k+1)g$.

For example, if $r = 3$, and $1g = 3$, $2g = 1$ and $3g = 2$, then any number of the form j, $j \in [n_1]$, maps to $n_1 + n_2 + j$. (We have used the notation $g \# s$ introduced in [ES], but our permutation is in fact the inverse of what Elgot and Shepherdson call $g \# s$. We think that this definition is slightly easier to decipher than theirs.) In [ES] it is proved that $g \# s$ is in fact a permutation.

LEMMA 2.5. *Let* $f = \alpha \cdot \langle a^\dagger, 1_p, \perp_{1,p} \rangle$ *and* $f' = \alpha' \cdot \langle a'^\dagger, 1_p, \perp_{1,p} \rangle$ *be* Σ*-terms* $n \to p$ *in s.n.f., where*

$$\alpha : n \to m + p + 1,$$

$$\alpha' : n \to m' + p + 1$$

$$a = \langle \sigma_1 \cdot \rho_1, ..., \sigma_t \cdot \rho_t \rangle : m \to m + p$$

$$a' = \langle \sigma'_1 \cdot \rho'_1, ..., \sigma'_{t'} \cdot \rho'_{t'} \rangle : m' \to m' + p,$$

where $\sigma_i \in \Sigma(r_i, s_i)$, $\rho_i : s_i \to m + p$, $i \in [t]$; *and where* $\sigma'_i \in \Sigma(r'_i, s'_i)$, ρ'_i: $s'_i \to m' + p$, $i \in [t']$.

Then $|f| = |f'|$ *iff both* $t = t'$ *and there is a permutation g of* $[t]$ *such that*

$$\text{(i)} \quad \sigma'_i = \sigma_{ig}, \; i \in [t],$$

$$\text{(ii)} \quad \rho'_i = \rho_{ig} \cdot ((g \# s)^{-1} + 1_p),$$

where s is the sequence $(r_1, ..., r_t)$, *and lastly*

$$\text{(iii)} \quad \alpha' = \alpha \cdot ((g \# s)^{-1} + 1_{p+1}).$$

Note that (i) *implies that* $r_{ig} = r'_i$ *and* $s_{ig} = s'_i$, *for* $i \in [t]$.

We omit the tedious proof.

It will be shown that the scheme algebras themselves are the free algebras in the equational class SCH generated by all algebras of the form Σ-Sch, as Σ varies over all ranked sets. We use this fact now to prove:

PROPOSITION 2.6. *The equational class generated by all scheme algebras is the same as the equational class generated by those scheme algebras* Σ*-Sch, for which* Σ *is a singly ranked set.*

Proof. Let Σ' be a doubly ranked set. We show that there is a singly ranked set Σ such that Σ'-Sch is isomorphic to a subalgebra of Σ-Sch. Indeed, define

$$\Sigma_p = \bigcup_n \Sigma'(n, p), \qquad \text{if } p \neq 1;$$

$$\Sigma_1 = \bigcup_n \Sigma'(n, 1) \cup \{\psi_1, \psi_2, ...\},$$

where $\{\psi_1, \psi_2, ...\}$ is an infinite set of *new* symbols $1 \to 1$.

Now we define a function $\varphi : \Sigma' \to \Sigma$-Sch as follows: for $\sigma \in \Sigma'(n, p)$, $n \neq 1$, let

$$\sigma\varphi = \langle \psi_1, ..., \psi_n \rangle \cdot \sigma.$$

When $n = 1$, $\sigma\varphi = \sigma$. It is easy to see that the induced homomorphism $\varphi^\# : \Sigma'$-Sch $\to \Sigma$-Sch is injective. Note that in the case $n = 0$, $\sigma\varphi = 0_1 \cdot \sigma$.

3. Axiomatizing Schemes

In this section we give a set of equational axioms whose models are the T_1 algebras in the equational class generated by the algebras Σ-Sch, as Σ ranges over all doubly ranked sets. The axioms are divided into two groups: SC_0, axioms for the operations not involving † (except for $\perp$), and axioms for †, SC_1. We let SC denote the union of SC_0 and SC_1. The equations are between terms in an appropriate term algebra Σ-TM. The letters f, g, h and k, sometimes with subscripts, denote atomic terms in Σ of the appropriate rank. The letters α and ρ denote simple total base terms, and φ, sometimes with subscripts, denotes a partial base term.

SC_0:

(a) $(f\cdot g)\cdot h=f\cdot(g\cdot h),\ f\colon n\to p,\ g\colon p\to q,\ h\colon q\to r;$

(b) $1_n\cdot f=f=f\cdot 1_p, f\colon n\to p;$

(c) $\langle\langle f, g\rangle, h\rangle=\langle f, \langle g, h\rangle\rangle, f\colon n\to p,\ g\colon m\to p,\ h\colon q\to p;$

(d) $\langle f, 0_p\rangle=f=\langle 0_p, f\rangle, f\colon n\to p;$

(e) $(f_1+f_2)\cdot\langle h_1, h_2\rangle=\langle f_1\cdot h_1, f_2\cdot h_2\rangle, f_i\colon n_i\to p_i,\ h_i\colon p_i\to q$

(f) $\langle f, g\rangle\cdot\varphi=\langle f\cdot\varphi, g\cdot\varphi\rangle, f\colon m\to p,\ g\colon n\to p,\ \varphi\colon p\to q;$

(g) $\langle\lambda, \kappa\rangle\cdot\langle f, g\rangle=\langle g, f\rangle, f\colon n\to p,\ g\colon m\to p;$

(h) $\pi_p^i\cdot\langle\varphi_1,\ldots, \varphi_p\rangle=\varphi_i, p>0,\ \varphi_j\colon 1\to q,\ j\in[p];$

(i) $\langle\pi_p^1\cdot\varphi,\ldots, \pi_p^p\cdot\varphi\rangle=\varphi,\ \varphi\colon p\to q.$

SC_1:

(a) $(0_n+f)^\dagger=f, f\colon n\to p;$

(b) $(f\cdot(1_n+g))^\dagger=f^\dagger\cdot g, f\colon n\to n+p,\ g\colon p\to q;$

(c) $\kappa\cdot\langle f, 0_n+g\rangle^\dagger=f^\dagger\cdot\langle g^\dagger, 1_p\rangle, f\colon n\to n+m+p,\ g\colon m\to m+p;$

(d) $\langle f\cdot(1_n+0_m+1_p),\ 0_n+g\rangle^\dagger=\langle f^\dagger, g^\dagger\rangle; f\colon n\to n+p,\ g\colon m\to m+p;$

(e) $\langle f, 0_n+g\rangle^\dagger=\langle f, 0_{n+m}+g^\dagger\rangle^\dagger, f\colon n\to n+m+p,\ g\colon m\to m+p;$

(f) $\langle f, 0_{n+m}+\varphi\rangle^\dagger=\langle(f\cdot(1_n+\langle\varphi, 1_p\rangle))^\dagger, \varphi\rangle, f\colon n\to n+m+p,\ \varphi\colon m\to p;$

(g) $(\alpha f^\dagger)^\dagger=(\alpha+0_m)\cdot\langle f, \alpha+0_{m+p}\rangle^\dagger, f\colon n\to n+m+p,\ \alpha\colon m\to n;$

(h) $\langle 1_n, \alpha\rangle\cdot(f\cdot(\langle 1_n, \alpha\rangle+1_p))^\dagger=\langle f, \alpha+0_{m+p}\rangle^\dagger, f\colon n\to n+m+p,\ \alpha\colon m\to n;$

(i) $(\rho\cdot f\cdot(\rho^{-1}+1_p))^\dagger=\rho\cdot f^\dagger, f\colon n\to n+p,\ \rho\colon[n]\to[n]$ a permutation.

The proof of the next lemma involves a tedious verification, and is omitted.

Lemma 3.1. *All identities in SC are valid in each scheme algebra.*

The following lemma is crucial for the axiomatization result. Its proof is in the Appendix. In the statement of this Lemma, Σ is an arbitrary doubly ranked set.

LEMMA3.2. *For any Σ-term $f: n \to p$ there is a Σ-term f',*

$$f' = \alpha \cdot \langle a^{\dagger}, 1_p, \perp_{1,p} \rangle$$

in s.n.f. such that the identity $f = f'$ is provable from SC.

The next theorem is one of our main results.

THEOREM 3.3. *If $f = f'$ is an equation such that $|f| = |f'|$ is true in Σ-Sch, then $f = f'$ is provable from SC.*

Proof. It is sufficient to show that for Σ-terms $f, f': n \to p$ in s.n.f.,

$$\mathrm{SC} \vdash f = f'$$

whenever $|f| = |f'|$. Let $f = \alpha \cdot \langle a^{\dagger}, 1_p, \perp_{1,p} \rangle$, and $f' = \alpha' \cdot \langle a'^{\dagger}, 1_p, \perp_{1,p} \rangle$, with $\alpha: n \to m + p + 1$, $a: m \to m + p$, $\alpha': n \to m' + p + 1$, $a': m' \to m' + p$. By Lemma 2.5, there is a permutation ρ: $[m] \to [m]$ such that

$$\mathrm{SC}_0 \vdash a' = \rho \cdot a \cdot (\rho^{-1} + 1_p),$$

$$\mathrm{SC}_0 \vdash \alpha' = \alpha \cdot (\rho^{-1} + 1_{p+1}).$$

By SC_1, (i)

$$\mathrm{SC} \vdash a'^{\dagger} = \rho \cdot a^{\dagger},$$

so that $\mathrm{SC} \vdash f = f'$.

COROLLARY 3.4. *Let $f = f'$ be an equation between Σ-terms. The following are equivalent.*

(i) *$f = f'$ is valid in Σ-Sch;*

(ii) *$|f| = |f'|$ in Σ-Sch.*

Proof. Clearly (i) implies (ii). Now suppose that (ii) holds, so that $f = f'$ is provable from SC. Hence (using the rule of substitution),

$$f\varphi = f'\varphi$$

is provable, for every endomorphism φ of the algebra of terms. Now if h: Σ-TM $\to$ Σ-Sch is any homomorphism, there is an endomorphism φ of Σ-TM such that for all terms t, $th = |t\varphi|$. Thus, $fh = |f\varphi| = |f'\varphi|$, by Lemma 3.1, and $|f'\varphi| = f'h$, completing the proof.

COROLLARY 3.5. *Let K denote the class of all algebras in which the equations SC are valid. Then Σ-Sch is freely generated in K by*

$$|\ |: \Sigma \to \Sigma - Sch.$$

Proof. Let $h: \Sigma\text{-TM} \to A$ be a homomorphism whose target is an algebra A in K. Then ker h, the congruence induced on Σ-Tm by h contains the set of equations SC, by definition of K. But then, h factors through $|\ |$, by Theorem 3.3 and Corollary 3.4.

4. Iteration Theories

In this section, we point out a connection between the axioms given above for the algebra of schemes, and a set of axioms for iteration theories. (We recall from [BEW] that an iteration theory is a preiteration theory which is a homomorphic image of a theory of trees which are the strong behaviors of Σ-schemes, for some Σ. Thus each iteration theory is also a homomorphic image of a scheme algebra Σ-Sch.) Let Th denote the set of axioms for theories given in Section 1.

Definition 4.1. IT is the set of axioms Th + SC_1 as well as the following two axiom schemes:

IT(a) $\lambda \cdot \langle f, 0_n + g\rangle^\dagger = g^\dagger, f: n \to n+m+p, g: m \to m+p;$

IT(b) (the "commutativity" axioms)

$$\langle \pi_m^1 \cdot \rho \cdot f \cdot (\rho_1 + 1_p), ..., \pi_m^m \cdot \rho \cdot f \cdot (\rho_m + 1_p)\rangle^\dagger = \rho \cdot (f \cdot (\rho + 1_p))^\dagger,$$

where $f: n \to m+p$, $\rho: [m] \to [n]$ is a total, surjective function, and $\rho_i: [m] \to [m]$ are total and satisfy $\rho_i \cdot \rho = \rho$.

Remark. IT(a) and SC_1(c) together yield

$$\langle f, 0_n + g\rangle^\dagger = \langle f^\dagger \cdot \langle g^\dagger, 1_p\rangle, g^\dagger\rangle.$$

IT is a good deal too generous: if $IT^\wedge$ denotes IT minus the axioms SC_1(d), (e), (f), (i), then it can be shown that $IT^\wedge$ is equivalent to IT. In fact, $IT^\wedge$ is essentially the axiom system appearing in [Es1]. Note that Th is strictly stronger than the set of axioms SC_0, since, for example, one cannot prove from SC_0 that

$$\pi_2^1 \cdot \langle f, g\rangle = f, \qquad \text{where } f, g: 1 \to 2 \text{ are not base schemes.}$$

Problem. What are the free algebras in the class of preiteration theories satisfying Th + SC_1?

Proposition 4.2. *Each identity in IT is valid in the class of all iteration theories.*

Proof. Since an iteration theory is a quotient of a theory, all of the axioms Th are valid. Since each iteration theory is a quotient of a scheme algebra, all of the axioms SC_1 are valid. It only remains to check the axioms IT(a) and (b). But both are easily seen to be valid.

In the remainder of this section, we outline a new proof that any identity valid in the class of all iteration theories is a consequence of the axioms IT. First, we observe that since each term $1 \to p$ denotes a scheme with one begin, we may call such a term "accessible" if each vertex in the corresponding scheme lies on a path from the begin. If t is a term of the form

$$\pi_n^i \cdot a^\dagger, \qquad \text{where } a = \langle \sigma_1 \cdot \rho_1, ..., \sigma_n \cdot \rho_n \rangle: n \to n+p \tag{$*$}$$

with each $\sigma_j \in \Sigma(1, r_j)$, and where $\rho_j: r_j \to n+p$ is a partial base term, then we may define the "behavior" of each vertex of the corresponding scheme in the obvious way, and call such a term 'reduced' if each vertex has a distinct behavior.

Starting from the scheme normal form, using the additional axioms, we may obtain a simpler normal form.

DEFINITION 4.3. Let Σ be a singly ranked set. A term $f: 1 \to p$ in Σ-TM_1 is in reduced normal form if one of the following conditions holds:

(i) f is π_p^i, some $i \in [p]$;

(ii) f is $\perp_{1,p}$, for some p;

(iii) f has the form $(*)$ above with each $\sigma_i \in \Sigma(1, r_i)$, and where $\rho_i: r_i \to n+p$ is a simple partial base term. Further, it is required that the scheme denoted by f is both accessible and reduced. (Thus, $\pi_n^i \cdot a^\dagger$ and $\pi_n^j \cdot a^\dagger$ are strongly equivalent iff $i=j$.)

Using Lemma 3.2, IT(a) and IT(b), one may prove:

LEMMA 4.4. *For each Σ-TM_1 term $f: 1 \to p$ there is a term g in reduced normal form such that $IT \vdash f = g$.*

The following fact was proved in [El77].

LEMMA 4.5. *Let f and f' be terms in reduced normal form $1 \to p$. If $f = f'$ is valid in all iteration theories, then either f is f' or*

$$f = \pi_n^i \cdot \langle \sigma_1 \cdot \rho_1, ..., \sigma_m \cdot \rho_m \rangle^\dagger,$$
$$f' = \pi_n^j \cdot \langle \sigma_1' \cdot \rho_1', ..., \sigma_n' \cdot \rho_n' \rangle^\dagger,$$

where $n = m$ and there is a bijection $g: [m] \to [m]$ with

$$\sigma_{kg}' = \sigma_k, \qquad \rho_{kg}' = \rho_k \cdot (g^{-1} + 1_p), \qquad \textit{all } k \in [m],$$
$$j = i(g^{-1} + 1_p).$$

Using these facts, we now prove:

THEOREM 4.6. *IT is a base of identities for iteration theories.*

Proof. One part of the proof is given by Proposition 4.2. Now assume that $f=f'$ is an identity valid in all iteration theories. By Lemma 4.4, we may assume that f and f' are both in reduced normal form. By Lemma 4.5 and $SC_1(i)$, $IT \vdash f=f'$, completing the proof.

APPENDIX

We now give the proof of Lemma 3.2, that every term is provably equal to one in s.n.f. from the axioms SC. For the reader's convenience, we repeat the axioms here.

SC_0:

(a) $(f \cdot g) \cdot h = f \cdot (g \cdot h), f: n \to p,\ g: p \to q,\ h: q \to r$;

(b) $1_n \cdot f = f = f \cdot 1_p, f: n \to p$;

(c) $\langle \langle f, g \rangle, h \rangle = \langle f, \langle g, h \rangle \rangle, f: n \to p, g: m \to p, h: q \to p$;

(d) $\langle f, 0_p \rangle = f = \langle 0_p, f \rangle, f: n \to p$;

(e) $(f_1 + f_2) \cdot \langle h_1, h_2 \rangle = \langle f_1 \cdot h_1, f_2 \cdot h_2 \rangle, f_i: n_i \to p_i, h_i: p_i \to q$

(f) $\langle f, g \rangle \cdot \varphi = \langle f \cdot \varphi, g \cdot \varphi \rangle, f: m \to p,\ g: n \to p,\ \varphi: p \to q$;

(g) $\langle \lambda, \kappa \rangle \cdot \langle f, g \rangle = \langle g, f \rangle, f: n \to p, g: m \to p$;

(h) $\pi_p^i \cdot \langle \varphi_1, ..., \varphi_p \rangle = \varphi_i, p > 0,\ \varphi_j: 1 \to q, j \in [p]$;

(i) $\langle \pi_p^1 \cdot \varphi, ..., \pi_p^p \cdot \varphi \rangle = \varphi,\ \varphi: p \to q$.

SC_1:

(a) $(0_n + f)^\dagger = f, f: n \to p$;

(b) $(f \cdot (1_n + g))^\dagger = f^\dagger \cdot g, f: n \to n+p, g: p \to q$;

(c) $\kappa \cdot \langle f, 0_n + g \rangle^\dagger = f^\dagger \cdot \langle g^\dagger, 1_p \rangle, f: n \to n+m+p, g: m \to m+p$;

(d) $\langle f \cdot (1_n + 0_m + 1_p), 0_n + g \rangle^\dagger = \langle f^\dagger, g^\dagger \rangle; f: n \to n+p, g: m \to m+p$;

(e) $\langle f, 0_n + g \rangle^\dagger = \langle f, 0_{n+m} + g^\dagger \rangle^\dagger, f: n \to n+m+p, g: m \to m+p$;

(f) $\langle f, 0_{n+m} + \varphi \rangle^\dagger = \langle (f \cdot (1_n + \langle \varphi, 1_p \rangle))^\dagger, \varphi \rangle, f: n \to n+m+p,\ \varphi: m \to p$;

(g) $(\alpha f^\dagger)^\dagger = (\alpha + 0_m) \cdot \langle f, \alpha + 0_{m+p} \rangle^\dagger, f: n \to n+m+p,\ \alpha: m \to n$;

(h) $\langle 1_n, \alpha \rangle \cdot (f \cdot (\langle 1_n, \alpha \rangle + 1_p))^\dagger = \langle f, \alpha + 0_{m+p} \rangle^\dagger, f: n \to n+m+p,\ \alpha: m \to n$;

(i) $(\rho \cdot f \cdot (\rho^{-1} + 1_p))^\dagger = \rho \cdot f^\dagger, f: n \to n+p,\ \rho: [n] \to [n]$ a permutation.

Remark 1. If φ and φ' are terms formed from the constants π_n^i and 0_p using composition, tupling *and dagger* which denote the same partial function $[n] \to [p]$, then the identity $\varphi = \varphi'$ is provable from SC. The iteration theory of partial functions is defined in [BEW, II, 3.8.1]. Indeed, one can prove by induction on the structure of each such term φ, there is a simple partial base term φ' such that $SC \vdash \varphi = \varphi'$. The only nontrivial case is to show, assuming that $\varphi: n \to n+p$ is a simple *pbt*, then one can find a simple *pbt* φ' such that $SC \vdash \varphi^\dagger = \varphi'$. The argument is by induction on n. In the case $n=0$, use $SC_0(i)$ (when $p=0$) to show

$SC \vdash \varphi = 0_p$, and also $SC_0 \vdash 0_p = 0_0 + 0_p$. Thus, by SC_1 (a), $SC \vdash \varphi^\dagger = 0_p$. In the case $n = 1$, there are three cases: $\varphi = \pi^1_{1+p}$, $\varphi = \pi^{1+i}_{1+p}$, for some $i > 0$, and $\varphi = \perp \cdot 0_{p+1}$. In the first case, we use SC_1(b); in the second and third, SC_1(a). In the induction step, one uses SC_1(i) and SC_1(d) (f) and (h). We omit the remaining details.

LEMMA 3.2. *For any Σ-term $f: n \to p$ there is a Σ-term f',*

$$f' = \alpha \cdot \langle a^\dagger, 1_p, \perp_{1,p} \rangle$$

in s.n.f. such that the identity $f = f'$ is provable from SC_0 and SC_1.

Proof. In our argument, we will make use of an intermediate normal form for terms.

DEFINITION. A term t is in (weak) normal form if

$$t = \alpha \cdot a^\dagger,$$

where α: $[n] \to [m]$ is a total function and a is a source tupling

$$a = \langle a_1, ..., a_s \rangle : m \to m + p,$$

where for each $i \in [s]$, a_i has either of the forms

$$\sigma \cdot \rho \qquad \text{or} \qquad \varphi \tag{$*$}$$

for some $\sigma \in \Sigma(r, q)$ and some total functions ρ: $[q] \to [m+p]$, and φ: $[1] \to [m+p]$. Below, we will refer to terms having one of the forms ($*$) as "primitive" terms.

Claim. For every term $f: n \to p$ there is a term $\alpha \cdot a^\dagger$ in normal form such that

$$SC \vdash f = \alpha \cdot a^\dagger.$$

Proof. By induction on the structure of the term f.

Case 1. f is $\sigma \in \Sigma(n, p)$.
Then $SC \vdash f = (0_n + \sigma)^\dagger$ and $SC \vdash (0_n + \sigma)^\dagger = 1_n \cdot \langle \sigma \cdot (0_n + 1_p) \rangle^\dagger$. Thus,

$$SC \vdash f = 1_n \cdot \langle \sigma \cdot (0_n + 1_p) \rangle^\dagger.$$

We now apply SC_0, ending Case 1.

Case 2. f is π^i_p, for some i and p.
Then SC_1, (a)$\vdash f = (0_1 + \pi^i_p)^\dagger$. But since $SC_0 \vdash (0_1 + \pi^i_p)^\dagger = 1_1 \cdot (\pi^{1+i}_{1+p})^\dagger$, we have

$$SC \vdash f = 1_1 \cdot (\pi^{1+i}_{1+p})^\dagger.$$

Case 3. f is 0_p. In this case one shows that $SC \vdash f = 1_0 \cdot 0_p{}^\dagger$.

Case 4. f is $f_1 \cdot f_2$, where $f_1 : n \to q$ and $f_2 : q \to p$. By the induction hypothesis, there are $\alpha_1 : [n] \to [m_1]$, $b_1 : m_1 \to m_1 + q$ and $\alpha_2 : [q] \to [m_2]$ and $b_2 : m_2 \to m_2 + p$ such that for $i = 1$ and 2,

$$SC \vdash f_i = \alpha_i \cdot b_i{}^{\dagger}.$$

Let $b = \langle b_1 \cdot (1_{m1} + \alpha_2 + 0_p), 0_{m1} + b_2 \rangle : m_1 + m_2 \to m_1 + m_2 + p$. We need the following fact.

Fact. $SC \vdash (1_n + 0_p) \cdot \langle f \cdot (1_n + h + 0_q), 0_n + g \rangle^{\dagger} = f^{\dagger} \cdot h \cdot g^{\dagger}$, for any $f : n \to n + m$, $g : p \to p + q$ and $h : m \to p$.

We omit the short proof, which uses SC_1, (b), (c) and SC_0. Using this fact. We get

$$SC \vdash b_1{}^{\dagger} \cdot \alpha_2 \cdot b_2{}^{\dagger} = (1_{m1} + 0_{m2}) \cdot b^{\dagger}.$$

One may show, using the Remark above and SC_0(f), that there is a term

$$a = \langle a_1, \ldots, a_t \rangle : m_1 + m_2 \to m_1 + m_2 + p$$

such that $SC_0 \vdash a = b$, and each a_i is primitive. But clearly then, $SC \vdash f_1 \cdot f_2 = (\alpha_1 + 0_{m2}) \cdot a^{\dagger}$, the proof of Case 4 is complete.

Case 5. f is $\langle f_1, f_2 \rangle$. Omitted.

Case 6. f is $f_1{}^{\dagger}$, where $f_1 : n \to n + p$. By the induction hypothesis, there is a term in normal form, $\alpha_1 \cdot b^{\dagger}$, where $\alpha_1 : [n] \to [m]$ and $b : m \to m + n + p$ such that $SC \vdash f_1 = a_1 \cdot b^{\dagger}$. By SC_1(g),

$SC \vdash f_1{}^{\dagger} = (\alpha_1 + 0_n) \cdot \langle b, \alpha_1 + 0_{n+p} \rangle^{\dagger}$. Thus this case is completed by SC_0. The claim is established.

End of Proof of Lemma 3.2. Let $f = \beta \cdot \langle b_1, \ldots, b_t \rangle^{\dagger}$, with $\beta : [n] \to [m]$, $\langle b_1, \ldots, b_t \rangle : m \to m + p$, and for each $i \in [t]$, b_i is primitive. By the permutation identity SC_1(i), we may assume that for some $k \in [t]$,

$$b_i = \sigma_i \cdot \rho_i \qquad \text{for } i \in [k]$$

and

$$b_i = \pi^j_{m+p} \qquad \text{for } i > k.$$

By $SC_1(h)$, we may further assume that when

$$b_i = \pi^{ji}_{m+p} \qquad \text{and} \qquad j_i = s \in [m]$$

then

$$b_s = \pi^{jg}_{m+p} \qquad \text{with } j_s \in [m],$$

so that $j_s > k$.

Let $c = \langle b_1, ..., b_k \rangle: r \to m+p$, $\varphi_0 = \langle b_{k+1}, ..., b_t \rangle: m-r \to m+p$. Then $SC_0 \vdash \varphi_0 = 0_r + \varphi$, for some total function $\varphi: [m-r] \to [(m-r)+p]$. By SC_1(e) and the Remark 1,

$$SC \vdash \langle c, \varphi_0 \rangle^\dagger = \langle c, 0_m + \varphi^\dagger) \rangle^\dagger = \langle c, 0_m + \rho \rangle^\dagger,$$

for some partial function $\rho: [m-r] \to [p]$. By SC_1(f),

$$SC \vdash \langle c, 0_m + \rho \rangle^\dagger = \langle \langle c \cdot (1_{m-r} + \langle \rho, 1_p \rangle))^\dagger, \rho \rangle.$$

Thus, for some simple total base term δ,

$$SC \vdash \langle c, 0_m + \rho \rangle^\dagger = (1_r + \delta) \cdot (c \cdot (1_{m-r} + \langle \rho, 1_p \rangle))^\dagger, 1_p, \perp_{1,p} \rangle,$$

and hence

$$SC \vdash f = \beta \cdot (1_r + \delta) \cdot (c \cdot (1_{m-r} + \langle \rho, 1_p \rangle))^\dagger, 1_p, \perp_{1,p} \rangle.$$

Put $\alpha = \beta \cdot (1_r + \delta)$. Since one may easily find a tupling of primitive terms a such that

$$SC \vdash c \cdot (1_{m-r} + \langle \rho, 1_p \rangle) = a,$$

and such that

$$\alpha \cdot \langle a^\dagger, 1_p, \perp_{1,p} \rangle$$

is in s.n.f., we have shown that

$$SC \vdash f = \alpha \cdot \langle a^\dagger, 1_p, \perp_{1,p} \rangle,$$

and the proof is complete.

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